Assignment 8.

This homework is due *Thursday*, October 26.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 7.

1. Quick reminder

Simple Approximation Lemma Let $f : E \to \mathbb{R}$ be measurable. Assume f is bounded on E. Then for each $\varepsilon > 0$, there are simple functions φ_ε, ψ_ε defined on E such that

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

Egoroff's Theorem Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Lusin's Theorem Let $f : E \to \mathbb{R}$ be measurable. Then for each $\varepsilon > 0$, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed set F contained in E such that

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

2. Exercises

(1) (3.2.12) Let f be a bounded measurable function on E. Show that there are sequences of simple functions on E, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing, $\{\psi_n\}$ is decreasing and each of these sequences converge to f uniformly on E. (*Hint:* Use Simple Approximation Lemma.)

(2) (3.2.21-) For a sequence $\{f_n\}$ of measurable functions with common domain E, show that $\inf\{f_n\}$ and $\sup\{f_n\}$ are measurable. (*Hint:* Express sup and inf as limits of appropriate sequences.)

(3) (3.3.26) Lusin's theorem provides that f equals to some continuous g at points of a set F. Must f itself be continuous at any point (as a function on E)?

(4) (3.3.27) Show that the conclusion of Egoroff's theorem is false if we drop the assumption that the domain E has finite measure.(*Hint:* Take a function f that is zero on $\mathbb{R}_-$ and nonzero on $\mathbb{R}_+$, consider $\{f(x-n)\}$.)

(5) (3.3.29) In-class proof of Lusin's theorem used finiteness of measure of E. Taking the case of finite measure for granted, prove Lusin's theorem for the case $m(E) = \infty$.

(6) (3.3.31) Let $\{f_n\}$ be a sequence of measurable functions on E that converges to the real-valued f pointwise on E. Show that $E = E_0 \cup \bigcup_{k=1}^{\infty} E_k$, where all sets $E_0, E_1, \ldots$ are measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 0$, and $m(E_0) = 0$.

3. Extra Exercises

Maybe coming in couple days.